



5.6 The Pythagorean theorem

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Introduction

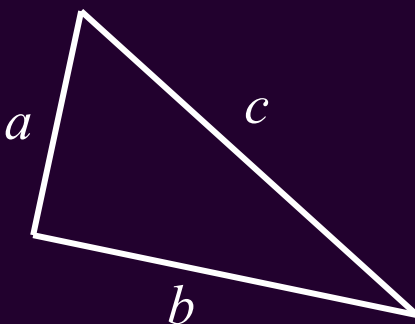
- In this topic, we will
 - Apply the Pythagorean theorem to vectors





The Pythagorean theorem

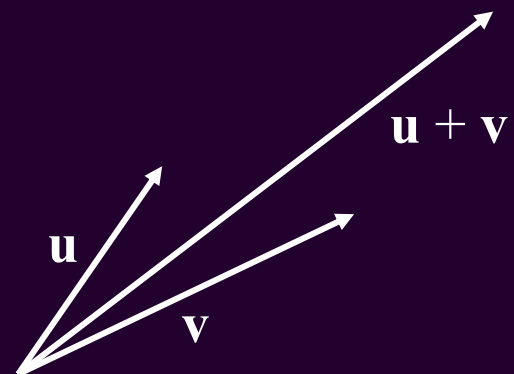
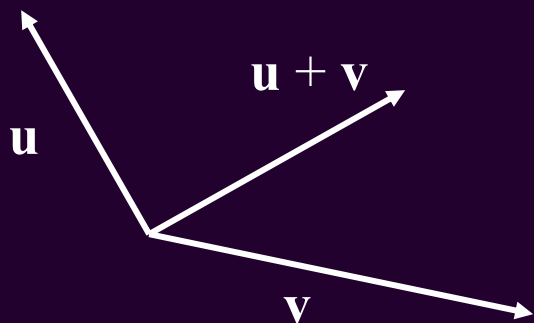
- You already know the Pythagorean theorem:
 - Given a right-angled triangle with sides of length a , b and c , then $a^2 + b^2 = c^2$





Orthogonality

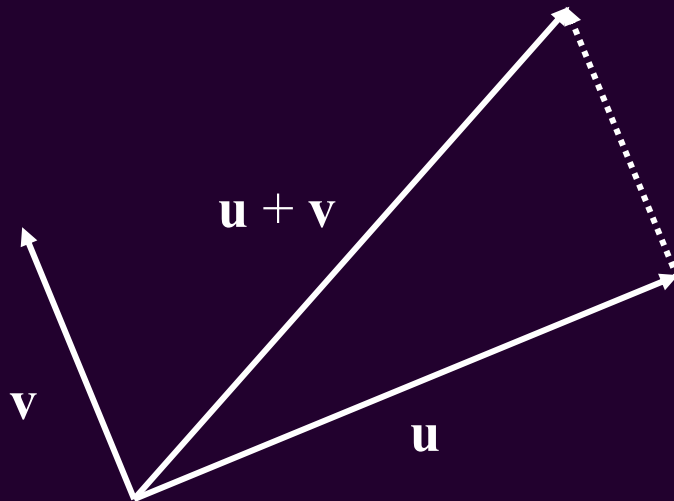
- Now, given two vectors $\mathbf{u}$ and $\mathbf{v}$,
is there a relationship between $\|\mathbf{u}\|_2$, $\|\mathbf{v}\|_2$, and $\|\mathbf{u} + \mathbf{v}\|_2$?





Orthogonality

- What if the two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal?



– Hypothesis: Is $\|\mathbf{u}\|_2^2 + \|\mathbf{v}\|_2^2 = \|\mathbf{u} + \mathbf{v}\|_2^2$?





Orthogonality

Theorem

If $\mathbf{u} \perp \mathbf{v}$, then $\|\mathbf{u}\|_2^2 + \|\mathbf{v}\|_2^2 = \|\mathbf{u} + \mathbf{v}\|_2^2$.

Proof:

Recall that $\|\mathbf{w}\|_2^2 = \langle \mathbf{w}, \mathbf{w} \rangle$,

$$\begin{aligned}\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle \\ = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle\end{aligned}$$

$$\begin{aligned}\text{Thus } \|\mathbf{u} + \mathbf{v}\|_2^2 &= \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle \\ &= \langle \mathbf{u}, \mathbf{u} + \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle \\ &= \langle \mathbf{u}, \mathbf{u} \rangle + \cancel{\langle \mathbf{u}, \mathbf{v} \rangle}^0 + \cancel{\langle \mathbf{v}, \mathbf{u} \rangle}^0 + \langle \mathbf{v}, \mathbf{v} \rangle \\ &= \|\mathbf{u}\|_2^2 + \|\mathbf{v}\|_2^2 \quad \blacksquare\end{aligned}$$





Orthogonality

- Thus, if the vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal:

$$\|\mathbf{u} + \mathbf{v}\|_2 = \sqrt{\|\mathbf{u}\|_2^2 + \|\mathbf{v}\|_2^2}$$

The diagram shows three vectors originating from a common point. Vector $\mathbf{u}$ points to the right, vector $\mathbf{v}$ points upwards, and vector $\mathbf{u} + \mathbf{v}$ points diagonally upwards and to the right. The length of vector $\mathbf{u}$ is labeled $\|\mathbf{u}\|_2$, the length of vector $\mathbf{v}$ is labeled $\|\mathbf{v}\|_2$, and the length of the resultant vector $\mathbf{u} + \mathbf{v}$ is labeled $\|\mathbf{u} + \mathbf{v}\|_2$. The vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular to each other.





Example

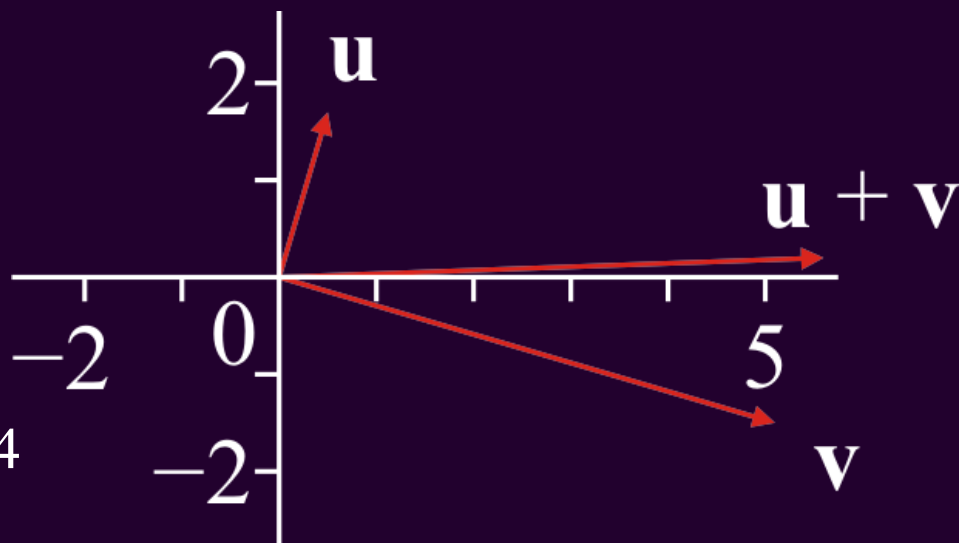
- For example, $\mathbf{u} = \begin{pmatrix} 0.5 \\ 1.7 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 5.1 \\ -1.5 \end{pmatrix}$ are orthogonal

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} 0.5 \\ 1.7 \end{pmatrix} + \begin{pmatrix} 5.1 \\ -1.5 \end{pmatrix} = \begin{pmatrix} 5.6 \\ -0.2 \end{pmatrix}$$

$$\|\mathbf{u}\|_2^2 = 0.5^2 + 1.7^2 = 3.14$$

$$\|\mathbf{v}\|_2^2 = 5.1^2 + 1.5^2 = 28.26$$

$$\|\mathbf{u} + \mathbf{v}\|_2^2 = 5.6^2 + 0.2^2 = 31.4$$





Summary

- Following this topic, you now
 - Understand the Pythagorean theorem applies to orthogonal vectors
 - Understand the proof
 - Have seen an example in $\mathbf{R}^2$





References

- [1] <https://en.wikipedia.org/wiki/Orthogonality>





Acknowledgments

None so far.





Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

<https://www.rbg.ca/>

for more information.





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